Optimum Code Rates for Noncoherent MFSK with Errors and Erasures Decoding over Rayleigh Fading Channels

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Abstract

In this paper, we analyze the performance of a communication system employing M-ary frequency shift, keying (FSK) modulation} with errors-and-crasutcs decoding using Viterbi ratio threshold technique for erasure insertion, in Rayleigh fading and AWGN channels. First, we maximize the code rate of the channel code that can be used reliably via the optimum use of erasures. This is accomplished by considering the capacity or the cutoff rate of the underlying discrete memoryless channel as the performance metric. Then, we examine the performance of the M-ary FSK system with optimum Reed-Solomon codes as measured by the minimum signal-to-noise ratio required to achieve a code word error probability of 10-5 via the optimum use of erasures.

1 Introduction

The use of error-correcting codes is of major importance in a variety of digital communication systems. Error-correcting codes arc particularly useful and can provide large potential gains in communication systems operating over pulsed interference channels or Rayleigh fading channels. For these channels, with forward error-correction, the improvement in signal-to-noise ratio can be on the order of 30 dB. This performance can be further improved if errors-all(l-cr~slltes decoding is employed in place of the usual hard decisions decoding.

in the literature, communication systems with errors-aml-erasures decoding have received a lot of attention. However, in these studies, the rate of the channel code is typically fixed and the code word error probability is evaluated for different methods of erasure insertion. For example, in [1], Baum and Pursley analyze the performance of frequency-]lop communications with Baysian erasure insertion; in [2], McKerracher and Wittke evaluate the performance of l?rcquclLcy-Hopped Spread Spectrum (FHSS) transmission with Reed-Solomon coding, parallel errors-and-erasures decoding and Viterbi ratio threshold technique. On the other hand, we find some studies investigating the implications for coding design in systems with errors-only decoding. For example, in [3], Stark computes the capacity and cutoff rate of noncoherent FSK with hard and soft decisions in Rician fading channels, from which optimum code rates arc determined; in [4], Khalona determines optimum Reed-Solomon codes for hard decision decoding of noncoherent FSK in Rician fading channels using the bit error probability as the performance criteria. Furthermore, in [5], Ritcey and Azizoğlu investigate the effect of blanking on the capacity and cutoff rate of a binary symmetric erasure channel. It is shown in this study that a controlled amount of erasures improves the channel capacity.

In this paper, wc analyze the performance of a communication system employing M-ary frequency shift keying (k'SK) modulation with errors-allcl-cras(lrcs decoding using Viterbi ratio thresholding for erasure insertion, over Rayleigh fading and AWGN channels. First, we maximize the code rate of the channel code that can be used reliably via the optimum usc of erasures. This is accomplished by considering the capacity and the cutoff rate of the underlying discrete memoryless channel as the performance metric. This code rate is optimum in the sense that it minimizes the signal-to-noise ratio necessary for reliable communications. !l'hen, wc examine the performance of the M-ary FSK system with optimum Reed-Solomon codes as measured by the minimum signal-to-noise ratio required to achieve a code word error probability of 10^{-5} via the optimum use of erasures.

2 System Description

We consider the system shown in Fig. 1. Data is first encoded with a nonbinary (N, K) Reed-Solomon code. The elements of the code words are selected from an alphabet of 2^q symbols, so that q information bits arc mapped into one of the 2^q symbols. The length of the code word is denoted by $N = 2^q - 1$, the number of information symbols encoded into a block of N symbols is denoted by $K = N - 2t$, where t symbol errors are correctable, and the normalized code rate is given by $r_c = K/N$.

A nonbinary Reed-Solomon code is particularly matched to an M-ary modulation scheme for

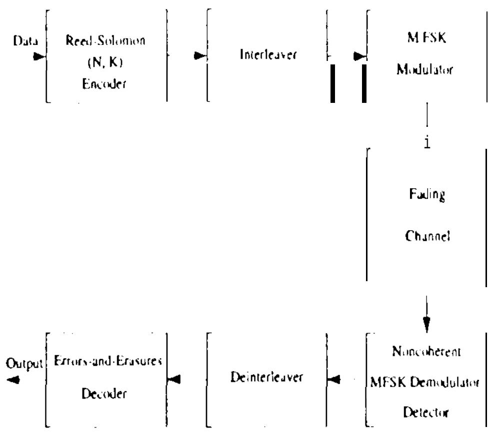

Figure 1: System diagram.

transmitting the $M = 2^q$ possible symbols. Specifically, M-ary FSK is frequently used, such that each of the 2^q symbols is mapped to one of the M orthogonal signals. ‘1’bus, the transmission of a code word is accomplished by transmitting a sequence of N orthogonal signals, where each signal is selected from the set of M possible signals. The sequence may be transmitted with or without interleaving. Interleaving enforces the memoryless channel under uncorrelated fading.

Therefore the modulation considered is that of M-ary FSK, with Viterbi ratio threshold test as the erasure generation technique, over slow nonselective Rayleigh fading (and AWGN) channels. The system is detected noncoherently, with the optimum detector basing its decisions on the squared envelopes of the received signal samples. Throughout the analysis, we assume that the channel is memoryless, which is equivalent to assuming that the attenuation due to fading is independent from symbol to symbol. In reality, this attenuation is a slowly varying function of time, meaning the real channel is not memoryless. The engineering solution to the memory in the channel is to employ au ideal interleaver.

Let us assume that the modulator and the demodulator/detector are included as part of the channel. This creates an equivalent discrete symmetric erasure channel with M inputs and $M+1$ outputs, since wc consider M-ary FSK modulation with erasure of unreliable symbols. If X is the input to the channel, thcu $X = x_k$ corresponds to transmitting a carrier modulated signal $s_k(t)$

$$s_k(t) = \Re\left\{s_{lk}(t)e^{j2\pi f_c t}\right\}, k = 1,2, \ldots, M, 0 \leq t \leq T, \tag{1}$$

where

$$s_{lk}(t) = \sqrt{\frac{2E_s}{T}}e^{j2\pi k\Delta f t}, \tag{2}$$

is the equivalent lowpass signal and Δf is the minimum frequency separation between adjacent frequencies. The M-ary orthogonal signals are equally probable and have equal energy E_s.

The received signal $r(t)$ consists of two components: the transmitted signal with attenuation α and phase ϕ uniformly distributed on the interval $[0, 2\pi]$, and a white Gaussian noise component

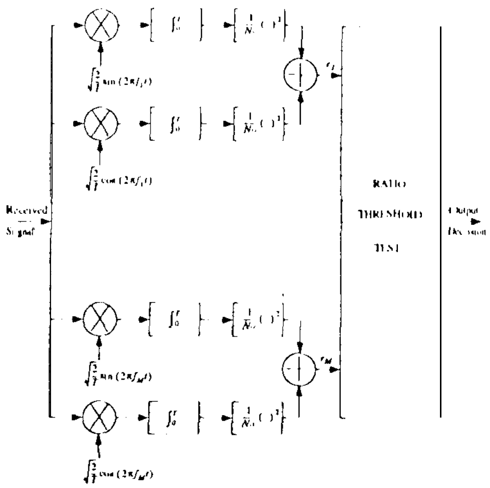

Figure 2: Receiver structure.

$n(t)$ with two-sided spectral density $N_0/2$. The attenuation α has mean square $E(\alpha^2) = 1$ and is Rayleigh distributed with

$$p_\alpha(x) = 2x\, e^{-x^2},\ x \geq 0. \tag{3}$$

Over one symbol interval, $t \in [0, T]$, the received signal is given by

$$r(t) = \Re\left\{\alpha e^{j\phi} s_{lk}(t) e^{j2\pi f_c t}\right\} + n(t). \tag{4}$$

The received signal is further passed through the demodulator and the noncoherent square-law detector. The optimum receiver structure which employs 2M correlators, two for each possible transmitted frequency $f_k = f_c + k\Delta f$, $k = 1, 2, \ldots$ M, is shown in Fig. 2. The M decision metrics at the detector are the M square-law envelopes,

$$r_k = \begin{cases} \frac{1}{N_0}|\sqrt{E_s}\alpha e^{j\phi} + n_1|^2 & k = 1 \\ \frac{1}{N_0}|n_k|^2 & k = 2, \ldots, M \end{cases} \tag{5}$$

The decision variables $\{r_k\}$ are then used as inputs to the decision device that employs Viterbi ratio threshold test [8]. Decisions on whether to erase are made by comparing envelope detector outputs to a fraction of the largest output in the set. More specifically, a received symbol is erased if

$$r_j > \lambda r_{i^*} \text{ for some } r_j \neq r_{i^*} \tag{6}$$

where $r_{i^*} = \max\{r_1, r_2, \ldots, r_M\}$ and λ is a fixed threshold in the interval [0, 1]. In other words, the gap between the largest and the second largest output in the set must be big enough for a symbol not to be erased. For those symbols *not* erased, standard hard-decision demodulation is employed. The decision device is followed by a Reed-Solomon errors-and-erasures decoder. Setting the erasure threshold $\lambda = 1$, results in errors-only decoding of the detected symbols.

Reed-Solomon codes are practically important for two reasons. First, Reed-Solomon codes arc maximum-distance separable, they have the largest possible 111111111111111 distance, i.e. $d_{\min} = N - K + 1$. Second, there exists efficient errors only and errors and erasures decoding algorithms, which make it possible to implement relatively long codes in many practical applications. A bounded-distance decoder capable of erasure decoding can correctly decode all received words containing s errors and e erasures as long as c and s satisfy the constraint (2s + c) $< d_{min}$. The performance of the bounded-distance errors-aud-crcasurcs decoder may be characterized by the code word error probability, P_e. This probability is equal to ouc minus the probability of correct decoding by the Reed-Solomondccodc!r, given by the tail of the trinomial

$$P_e = 1 - \sum_{s=0}^{\lfloor \frac{d_{\min}-1}{2} \rfloor} \sum_{e=0}^{d_{\min}-2s-1} \binom{N}{s} \binom{N-s}{e} p_e^e p_s^s (1 - p_e - p_s)^{N-s-e}, \tag{7}$$

where p_s, p_e is the probability of error aud erasure, respectively.

3 Performance Analysis

In this section we compute the probabilities of correct symbol decision, p_c, symbol erasure, p_e and incorrect symbol decision, p_s for the noncoherent M-ary FSK system with Viterbi ratio threshold technique for erasing unreliable symbols, in Rayleigh fading and AWGN channels. Since transmitted symbols arc assumed to be equally probable, it is sufficient to calculate p_c, p_e, and p_s when the symbol x_1 is transmitted. Wc compute these probabilities following Baum and Pursley [1]. Let $f(\cdot)$ aud $F(\cdot)$ denote the pdf aud cdf of the $\{r_i\}$'s containing signal plus noise, aud $g(\cdot)$ aud $G(\cdot)$ be the pdf and cdf of the $\{r_j\}$'s containing the noise alone. The probability of a correct symbol decision, p_c, is given by

$$\begin{aligned} p_c &= \Pr[r_1 > r_j, \lambda r_1 > r_j \text{ for all } j \# 1] \\ &= \Pr[\lambda r_1 > r_j \text{ for all } j \neq 1], \end{aligned} \tag{8}$$

where the second equality follows from the fact that $0 \leq \lambda \leq 1$. Conditioning on the event $\{r_1 = x\}$, Wc find

$$\begin{aligned} p_c &= \int_0^{\infty} [G(\lambda x)]^{M-1} f(x) dx \\ &= \int_0^{\infty} \left[\int_0^{\lambda x} g(y) dy \right]^{M-1} f(x) dx. \end{aligned} \tag{9}$$

The probability of incorrect symbol decision, p_s, is equal to $(M-1)$ times the channel transition probability that, say y_2 is the output given that x_1 is the channel input

$$\begin{aligned} \Pr(y_2|x_1) &= \Pr[r_2 > r_j, \lambda r_2 > r_j \text{ for all } j \# \text{-} 2] \\ &= \Pr[\lambda r_2 > r_1, \lambda r_2 > r_j \text{ for all } j \neq 1, 2] \end{aligned} \tag{10}$$

Conditioning on the event $\{r_2 = x\}$ and then averaging over the values of x gives

$$
\begin{aligned}
p_s &= (M-1)\int_0^{\infty} [F(\lambda x)]\,[G(\lambda x)]^{M-2}\, g(x)dx \\
&= (M-1)\int_0^{\infty} \left[\int_0^{\lambda x} f(y)dy\right] \left[\int_0^{\lambda x} g(y)dy\right]^{M-2} g(x)dx.
\end{aligned} \tag{11}
$$

The probability of symbol erasure can be found from the expression

$$p_e = 1 - p_c - p_s\,. \tag{12}$$

In presence of Rayleigh fading, the $\{r_k\}$'s are exponentially distributed, Proakis [6], with

$$f(r) = \frac{1}{1+\Gamma}\exp\left\{-\frac{r}{1+\Gamma}\right\}, \tag{13}$$

and

$$F(r) = 1 - exp\left\{-\frac{r}{1+\Gamma}\right\}. \tag{14}$$

Here $\Gamma = E_s/N_0$ is the average received symbol signal-to-noise ratio. In the case of AWGN and without fading, the $\{r_k\}$'s are non-central chi-square distributed. The corresponding pelf and cdf are given by

$$f(r) = \exp\{-(r+\Gamma)\}\, I_0\left(2\sqrt{r\Gamma}\right), \tag{15}$$

$$F(r) = 1 - Q_1\left(\sqrt{2\Gamma}, \sqrt{2r}\right). \tag{16}$$

Here I_0 is the modified Bessel function and Q_1 is Marcum's Q-function [6, p.44]. When no signal is present, the $\{r_k\}$'s are exponentially distributed in both the Rayleigh fading and AWGN case. By virtue of our scaling, the pdf and cdf are given by

$$g(r) = \exp\{-r\}, \tag{17}$$

$$G(r) = 1 - \exp\{-r\}. \tag{18}$$

For Rayleigh fading channels, closed-form expressions for the probability of correct symbol decision and the probability of symbol erasure are obtained using binomial expansion

$$p_c = \sum_{j=0}^{M-1} (-1)^j \binom{M-1}{j} \frac{1}{j\lambda(1+\Gamma)+1}, \tag{19}$$

$$p_s = \sum_{j=0}^{M-2} (-1)^j \binom{M-1}{j+1} \frac{\lambda(j+1)}{\lambda(j\lambda+1)+(1+\Gamma)(j\lambda+1)^2}. \tag{20}$$

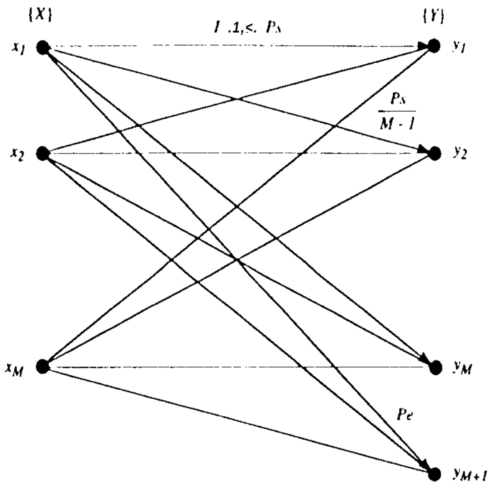

Figure 3: M-ary Discrete Symmetric Channel with Erasure.

However, for $M \geq 32$, (19) and (20) are numerically ill conditioned and we compute p_s and p_c by Gauss Laguerre integration of order zero.

For AWGN channels, no closed-form expressions are known. The probability of correct symbol decision and the probability of symbol erasure are computed numerically from

$$p_c = \int_0^{\infty} \left[1 - e^{-\lambda x}\right]^{M-1} e^{-(x+\Gamma)} I_0\left(2\sqrt{\Gamma x}\right) dx, \tag{21}$$

$$p_s = (M-1) \int_0^{\infty} \left[1 - Q_1\left(\sqrt{2\Gamma}, \sqrt{2\lambda x}\right)\right] \left[1 - e^{-\lambda x}\right]^{M-2} e^{-x} dx \tag{22}$$

4 Capacity and Cutoff Rate

The underlying channel model of the M-ary system described in section 2, results in the discrete symmetric channel with erasure illustrated in Fig. 3. The input-output characteristics of the channel are described by the set of transition probabilities $\{p_{ij}\}$:

$$p_{ij} = \Pr(y_j|x_i) = \begin{cases} 1 - p_s - p_e & i = j \qquad i = 1, 2, \ldots, M \\ p_s/(M-1) & i \neq j \qquad j \neq M+1 \\ p_e & j = M+1 \end{cases} \tag{23}$$

where p_e is the probability of symbol erasure and p_s is the probability of incorrect symbol decision computed in section 3.

The capacity of this discrete memoryless channel with a finite input alphabet $X = \{x_1, x_2, \ldots, x_M\}$ and a finite output alphabet $Y = \{y_1, y_2, \ldots, y_M, y_{M+1}\}$, where y_{M+1} is an erasure symbol, is given by [7, p.74].

$$C = \max_{\Pr(x_i)} I(X;Y), \tag{24}$$

where $I(X;Y)$ is the average mutual information provided by the output Y about the input X. Due to the symmetry of the channel, the distribution that achieves capacity is the uniform distribution $\Pr(x_i) = 1/M$, for $i = 1, 2, \ldots, M$. After simplification, the capacity of the M-ary symmetric channel with erasure is found to be

$$C = \log_2 M (1-p_e)\left(1 - H_M\left(\frac{p_s}{1-p_e}\right)\right) \quad \text{bits/channel use} \tag{25}$$

where $H_M(\cdot)$ is the M-ary entropy function

$$H_M(x) = -x \log_M(x) - (1-x)\log_M(1-x) + x\log_M(M-1). \tag{26}$$

For the same channel model, we define the cutoff rate as

$$R_0 = \max_{\Pr(x_i)} \left\{ -\log_2 \sum_{j=1}^{M+1} \left[\sum_{i=1}^{M} \Pr(x_i)\Pr(y_j|x_i)^{1/2} \right]^2 \right\}. \tag{27}$$

As in the case of channel capacity, an equiprobable distribution on the input alphabet, $\Pr(x_i) = 1/M$, for $i = 1, 2, \ldots, M$ achieves the maximum value. Consequently, under the uniform input distribution, the expression for R_0 reduces to

$$R_0 = \log_2 M - \log_2\left[1 + (M-1)p_e + (M-2)p_s + 2\sqrt{(M-1)p_s(1-p_s-p_e)}\right]. \tag{28}$$

The expressions given above in equation (25) and (28) are used together with p_e and p_s from section 3 to obtain the capacity and cutoff rate of the coded system. The results are presented in the next section.

5 Results

In this section we present some numerical results for the channel capacity and cutoff rate. We are interested in finding the largest possible transmission rate for which reliable communication is possible, or equivalently, the smallest possible information bit signal-to-noise ratio, $\mathcal{E}_b/N_0$, which guarantees arbitrarily small error probability with codes of rate R_c (information bits per channel symbol). The channel coding theorem of information theory guarantees that there exist channel codes (and decoders) that make it possible to achieve reliable communication, with as small an error probability as desired, if the transmission rate is less than the capacity. We take cutoff rate, R_0, as a practical value of capacity as suggested by the computational complexity of sequential decoders [10, p.318]. Therefore, we calculate the minimum $\mathcal{E}_b/N_0$ necessary for reliable communications by computing the channel capacity and cutoff rate. Errors-and-erasures decoding is assumed, and the results are obtained by optimizing the threshold λ to maximize C and R_0. We compare the results for the Rayleigh fading channels with those for the AWGN channel. Both errors only and errors and erasures is considered.

We follow the analysis presented by Stark [3] and Wilson [10]. First, we relate the channel symbol signal-to-noise ratio, E_s/N_0, to the information bit signal-to-noise ratio, $\mathcal{E}_b/N_0$ by $E_s/N_0 =$

$R_c\mathcal{E}_b/N_0$. Next, we equate $R_c = C$ (or $R_c = R_0$), and find the solution for $\mathcal{E}_b/N_0$ for different code rates in the range $0 < R_c < \log_2 M$. This solution provides a lower bound on $\mathcal{E}_b/N_0$ required for reliable communication. From (25) and (28), we see that for a given alphabet size M, the capacity and cutoff rate of the discrete symmetric channel with erasure are functions of p_s, the symbol error probability, and p_e, the symbol erasure probability, which in turn depend on $R_c\,\mathcal{E}_b/N_0$ and λ. To remove the λ dependence, we maximize over $0 \leq \lambda \leq 1$ to obtain

$$R_c = C\left(R_c\frac{\mathcal{E}_b}{N_0}, \lambda^*\right), \tag{29}$$

and solve numerically for

$$\frac{\mathcal{E}_b}{N_0} = \frac{C^{-1}(R_c, \lambda^*)}{R_c}, \tag{30}$$

where λ^* is the optimum erasure threshold. Similarly, based on cutoff rate calculations, the lower bound on $\mathcal{E}_b/N_0$ required for reliable communication is found by solving

$$\frac{\mathcal{E}_b}{N_0} = \frac{R_0^{-1}(R_c, \lambda^*)}{R_c} \tag{31}$$

Both capacity and cutoff rate are functions of the erasure threshold λ. In Fig. 4 we illustrate the channel capacity (in bits per channel use) with Rayleigh fading for different values of the alphabet size M, at $E_b/N_0 = E_s/N_0 \log_2 M = 8$ dB. Notice that there exists an optimum erasure threshold λ^* which maximizes the channel capacity, and that λ^* varies with M. Observe that for a constant $E_b/N_0, \lambda^*$ increases with M.

The locus of the solutions to (30) and (31) are illustrated in Fig. 5 for channel capacity, and Fig. 6 for cutoff rate. These loci were obtained by picking E_s/N_0 values, finding the maximum capacity (or cutoff rate) with optimum threshold selection, equating $R_c = C$ (or R_0), and then converting E_s/N_0 to the required $\mathcal{E}_b/N_0$. In Fig. 5 we show the minimum $\mathcal{E}_b/N_0$ loci implied by the capacity limit for noncoherent demodulation of binary FSK on the interleaved Rayleigh channel and the AWGN channel. In addition, we show the minimum $\mathcal{E}_b/N_0$ loci of the solutions to equation (30) for $\lambda = 1$, corresponding to the errors-only decoding case. Similarly, in Fig. 6 we illustrate the minimum $\mathcal{E}_b/N_0$ loci implied by the cutoff rate limit, for the cases of errors-and-erasures decoding and errors-only decoding, in fading and nonfading channels.

We observe several interesting features from the curves of Figs. 5 and 6. First, we note that there exist optimum rates in the minimum energy sense, for all the cases shown. Thus, the information bit signal-to-noise ratio $\mathcal{E}_b/N_0$ reaches a minimum at the optimum code rate, and then increases for both higher and lower rates. This behavior is not found with coherent demodulation, in which case $\mathcal{E}_b/N_0$ decreases monotonically with a decrease in code rate [6, p.404]. Secondly, we observe that errors-and-erasures decoding with optimum threshold selection is more efficient in the minimum energy sense than errors-only decoding for both Rayleigh and AWGN channels. This is a main conclusion of our study. For example, if the optimal code rate is chosen based on cutoff rate for the Rayleigh channel, errors-and-erasures decoding costs around 0.7 dB less relative to the minimum $\mathcal{E}_b/N_0$ necessary for errors-only decoding.

Table 1: Minimum $\mathcal{E}_b/N_0$ and optimum code rates based on capacity.

		λ	$\mathcal{E}_b/N_0$ (dB)	Rate
Rayleigh	Errors-and-erasures	0.49	9.62	0.23
	Errors-only	1.00	10.24	0.'21
AWGN	Errors-and-erasures	0.55	7.34	0.52
	Errors-only	1.00	7.82	0.52

Table 2: Minimum $\mathcal{E}_b/N_0$ and optimum code rates based on cutoff rate.

		λ	$\mathcal{E}_b/N_0$ (dB)	Rate
Rayleigh	Errors-and-erasures	0.45	12.21	0.16
	Errors-only	1.00	12.94	0.13
AWGN	~~13rrors-and~~-erasures	0.50	9.16	0.54
	Errors-only	1.00	9.84	0.46

Lastly, wc notice that the AWGN channel exhibits a broad optimum-rate region from about 0.3 to 0.7 bits/channel use, while the Rayleigh channel is characterized by smaller optimum rates and greater sensitivity to the code rate variations. Also, note that the loss incurred by Rayleigh fading compared to no fading, in the capacity sense, is around 2.3 dB for the optimal code rate and errors-at@cm-surcs decoding. The optimum code rates and erasure thresholds together with the minimum $\mathcal{E}_b/N_0$ based on channel capacity and cutoff rate for the binary noncoherent FSK are summarized in Tables 1 and 2 respectively.

For M-ary noncoherent FSK with M larger than 2, the minimum information bit signal-to-noise ratio implied by the capacity limit is shown in Fig. 7 for the Rayleigh channel, and in Fig. 8 for the AWGN channel. In both figures, assuming errors-and-erasures decoding with optimum threshold selection, we plot the minimum $\mathcal{E}_b/N_0$ loci versus the normalized code rate, $r_c = R_c/\log_2 M$ (in information symbols per channel use), for $M =$ 2,4,8, and 16. Two equivalent plots for the minimum information bit signal-to-mist ratio implied by the cutoff rate are illustrated in Fig. 9 for the Rayleigh channel, and in Fig. 10 for the AWGN channel. From all these plots we observe that as M increases beyond 2, the minimum $\mathcal{E}_b/N_0$ needed for reliable communications reduces.

Finally, wc determine how well the optimum code rates obtained from the channel capacity and cutoff rate limit predict the performance of practical, finite-length Rind-Solomon codes. Consequently, we evaluate the performance of Reed-Solomon coded M-ary FSK modulation with errors-and-(!rasurcs decoding over the Rayleigh and AWGN channels. Assuming codes of length $N = M - 1$, the performance of the optimum (N, K) Reed-Solomon codes can be determined using (7) for the probability of code word error P_e.

In Fig. 11 we illustrate the $\mathcal{E}_b/N_0$ loci measured to achieve a code word error probability of 10^{-5}, for M = 16,32,64,128, and 256 in Rayleigh fading channels. Similar curves are shown in Fig. 12 for AWGN channels. These curves were obtained by varying the information symbols into the code and therefore the code rate, and at each rate finding numerically the signal-to-noise ratio with optimum threshold selection such that $P_e = 10^{-5}$. We notice that these plots have similar characteristics as those based on capacity or cutoff rate limit, and that they essentially predict

Table 3: Optimum Reed-Solomon codes and $\mathcal{E}_b/N_0$ required for P_e = 10^{-5}.

Rayleigh			AWGN		
λ	(N, K)	$\mathcal{E}_b/N_0$ (dB)	λ	(N, K)	$\mathcal{E}_b/N_0$ (dB)
0.4	(15, 3)	14.7	0.7	(15,9)	7.1
0.5	(31, 7)	11.5	0.8	(31,21)	5.6
0.6	(63, 17)	9.2	0.8	(63,43)	4.5
0.6	(127, 35)	7.8	0.8	(127, 83)	3.8
0.7	(255, 73)	6.8	0.9	(255,173)	3.2

the same optimum code rates. Also, it is observed that the signal-to-noise ratio required for a 10^{-5} code word error probability with short length Reed-Solomon codes is much higher than the lower limit on $\mathcal{E}_b/N_0$ for reliable communication. However, as the code word length (equivalently the signal set size) increases, the gap between the signal-to-noise ratio required for a 10^{-5} code word error probability and the lower limit predicted by the capacity (or cutoff rate) decreases. For example, with M = 256, the $\mathcal{E}_b/N_0$ required by Reed-Solomon codes is within approximately 2.8 dB of the capacity limit in Rayleigh channels, and within 1.2 dB of the capacity limit in AWGN channels. The optimum Reed-Solomon codes, together with the information bit signal-to-noise ratio $\mathcal{E}_b/N_0$ required to achieve $P_e = 10^{-5}$, and the corresponding erasure thresholds are summarized in Table 3. Moreover, our results show a significant performance improvement with errors-and-erasures decoding of the optimum Reed-Solomon codes listed in Table 3, compared to errors-only decoding of the same codes in Rayleigh channels. For example, at a code word error probability of 10^{-5}, errors-only decoding requires at least 3 dB more than errors-and-erasures decoding of the (15, 3) Reed-Solomon code in Rayleigh fading. On the other hand, there is very little performance improvement with errors-and-erasures decoding over errors-only decoding for the AWGN channels.

6 Conclusion

The purpose of this paper is to design error-correcting codes for an M-ary noncoherent FSK system with errors-and-erasures decoding, on channels subjected to Rayleigh fading (or white Gaussian noise). We show that Reed-Solomon codes can be designed for optimum rate and optimum erasure threshold, as measured by the minimum signal-to-noise ratio necessary for reliable communications. This minimum is calculated by computing the channel capacity or cutoff rate. Using the channel capacity (or cutoff rate) as the performance criteria is justified by the channel coding theorem of information theory, which guarantees the existence of codes with rates less than the capacity, for which arbitrarily small error probability is possible. Moreover, we are interested in comparing the performance of these infinite length codes, with the performance of practical errors-and-erasures Reed-Solomon codes for the M-ary FSK system. Optimum Reed-Solomon codes were obtained to minimize the signal-to-noise ratio necessary to achieve a probability of code word error of 10^{-5}. It was shown that the results obtained for optimum rates of Reed-Solomon codes are close to those obtained by the capacity or cutoff limit. The main conclusions drawn from our results are:

- Ijow-rate codes (in information symbols/channel USC) are optimum for the M-ary noncoherent FSK system over Rayleigh fading channels, provided significant bandwidth expansion is acceptable. In AWGN channels, moderate-rate codes are optimum for the M-ary noncoherent FSK system.

- The energy efficiency of the M-ary noncoherent FSK system is increasing with a larger alphabet size, at the expense of reduced bandwidth efficiency.

- The optimum code rates predicted by the capacity or cutoff rate limit are very close to the optimum rates of Reed-SolomoI~ codes achieving a code word error probability of 10^{-5} with the M-ary noncoherent system.

- Erasures decoding of Reed-Solornon codes provides significant performance improvement over the fading channels (measured by the probability of code word error), but does not provide significant gain for the additive white Gaussian noise channels.

The results obtained here can serve as benchmarks of obtainable performance and should be useful in validating the results of simulation studies. Furthermore, the analysis presented can be readily extended to other fading channel models, such as the Rician and Nakagami fading models.

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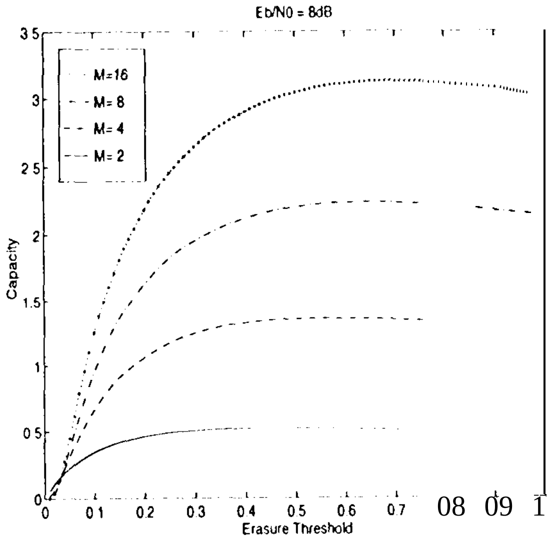


Figure 4: Capacity versus erasure threshold for M = 2,4,8,16 at $E_b/N_0 = 8$ dB, Rayleigh channel.

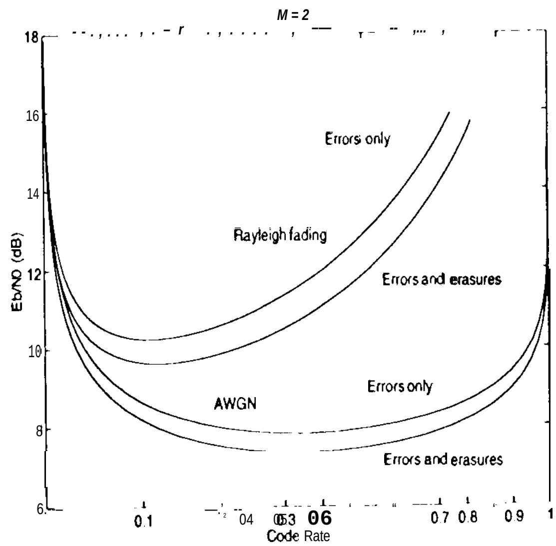
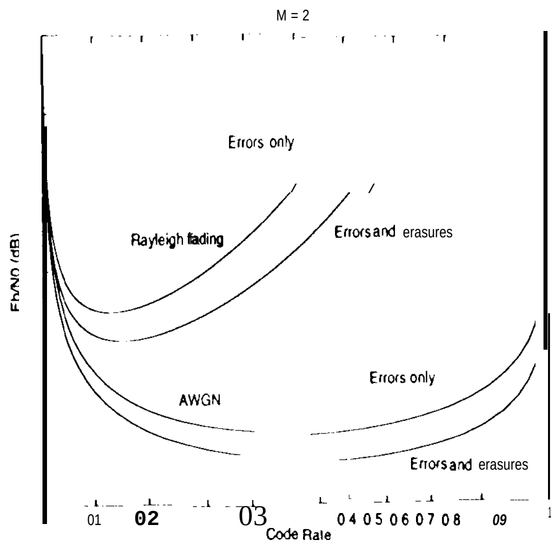


Figure 5: $\mathcal{E}_b/N_0$ loci based on the capacity, $M = 2$.



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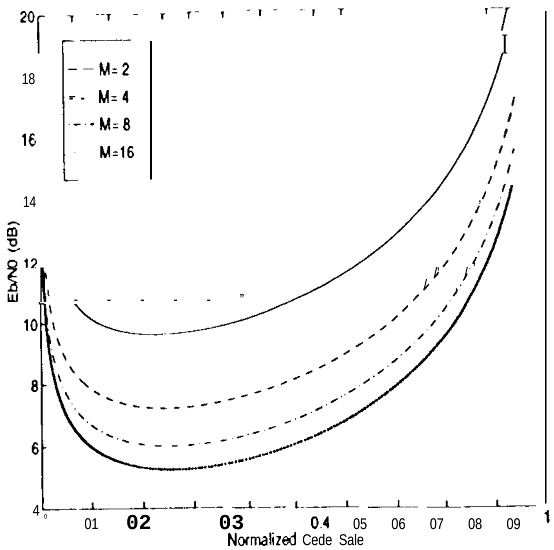
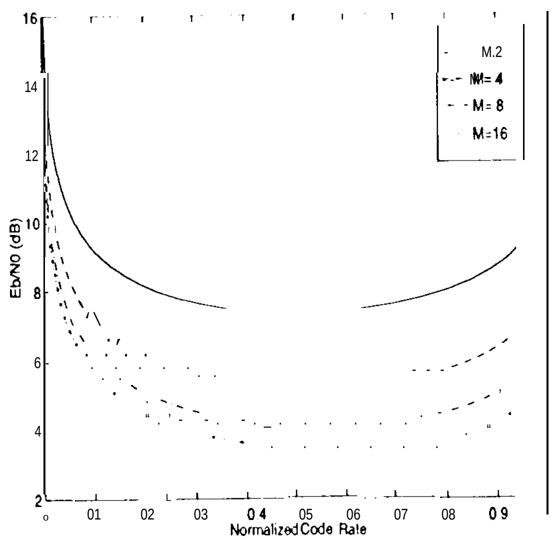


Figure 7: Minimum $\mathcal{E}_b/N_0$ loci based on capacity, Rayleigh channel.



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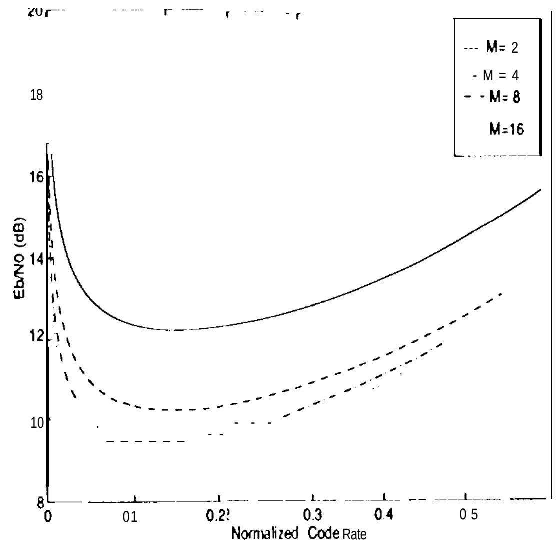

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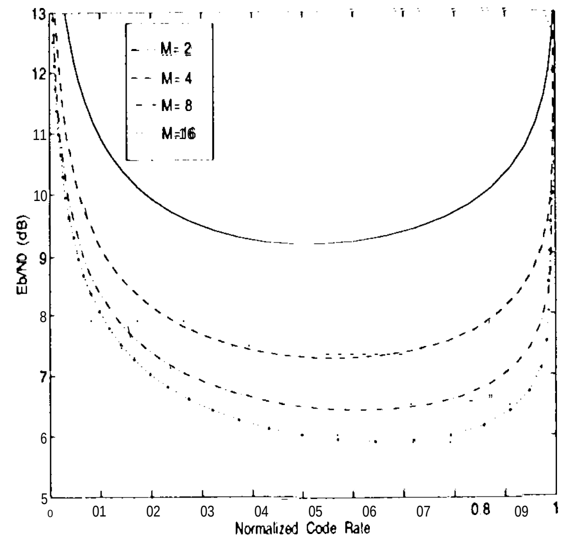

Figure 10: Minimum $\mathcal{E}_b/N_0$ loci based on cutoff rate, AWGN channel.